Filtrage, estimation et feedback pour les systèmes quantiques ouverts

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Outline

Models of open quantum systems

- The Markov chain describing the LKB photon-box
- From discrete-time to continuous-time models
- Key role of spin/spring composite systems

Mathematical **quantum** systems theory

- Two kind of feedback for open quantum systems
- Quantum filtering and estimation
- And many other subjects

Three quantum features[1]

1. Schrödinger equation: wave function $|\psi\rangle \in \mathcal{H}$, density operator ρ

$$\frac{d}{dt}|\psi\rangle = -iH|\psi\rangle, \quad \frac{d}{dt}\rho = -i[H, \rho]$$

2. Origin of dissipation and irreversibility: collapse of the wave packet induced by the measure of observable $\mathcal{O}$ with spectral decomposition $\sum_\mu \lambda_\mu P_\mu$:
   - measure outcome λ_μ with proba. $p_\mu = \langle\psi|P_\mu|\psi\rangle = \text{Tr}\left(\rho P_\mu\right)$ depending on $|\psi\rangle$, ρ just before the measurement
   - measure back-action if outcome λ_μ:

$$|\psi\rangle \mapsto |\psi\rangle_+ = \frac{P_\mu|\psi\rangle}{\sqrt{\langle\psi|P_\mu|\psi\rangle}}, \quad \rho \mapsto \rho_+ = \frac{P_\mu \rho P_\mu}{\text{Tr}\left(\rho P_\mu\right)}$$

3. Tensor product for the description of composite systems (S, M):
   - Hilbert space $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_M$
   - Hamiltonian $H = H_S \otimes \mathbb{I}_M + H_{int} + \mathbb{I}_S \otimes H_M$
   - observable on sub-system M only: $\mathcal{O} = \mathbb{I}_S \otimes \mathcal{O}_M$.

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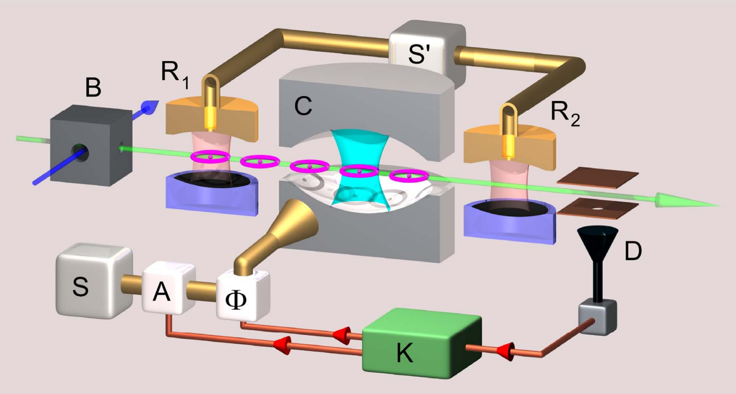
[1] S. Haroche and J.M. Raimond. *Exploring the Quantum: Atoms, Cavities and Photons.* Oxford Graduate Texts, 2006.

LKB photon Box: $\mathcal{H}_S$ cavity, $\mathcal{H}_M$ flying atom.

[2]



The experiment measuring and controlling the photons trapped inside the cavity C (measurement-based feedback).

[2] C. Sayrin, I. Dotsenko, X. Zhou, B. Peaudecerf, Th. Rybarczyk, S. Gleyzes, P. Rouchon, M. Mirrahimi, H. Amini, M. Brune, J.M. Raimond, S. Haroche: Real-time quantum feedback prepares and stabilizes photon number states. Nature, 477(7362), 1 September 2011.

The Markov chain model (1)

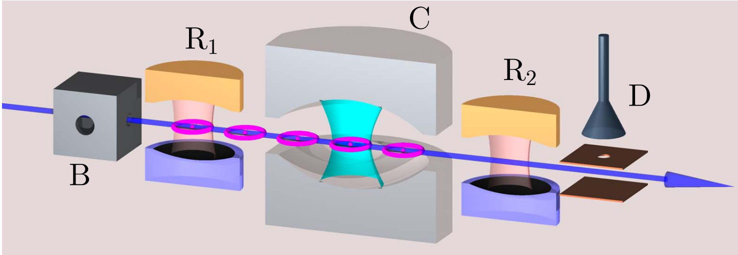
- System S corresponds to a quantized mode in C:

$$\mathcal{H}_S = \left\{ \sum_{n=0}^{\infty} \psi^n |n\rangle \mid (\psi^n)_{n=0}^{\infty} \in l^2(\mathbb{C}) \right\},$$

  where $|n\rangle$ represents the Fock state associated to exactly n photons inside the cavity
- Meter M is associated to atoms : $\mathcal{H}_M = \mathbb{C}^2$, each atom admits two energy levels and is described by a wave function $c_g|g\rangle + c_e|e\rangle$ with $|c_g|^2 + |c_e|^2 = 1$; atoms leaving B are all in state $|g\rangle$
- When atom comes out B, the state $|\Psi\rangle_B \in \mathcal{H}_S \otimes \mathcal{H}_M$ of the composite system atom/field is separable

$$|\Psi\rangle_B = |\psi\rangle \otimes |g\rangle.$$

The Markov chain model (2)



- When atom comes out B: $|\Psi\rangle_B = |\psi\rangle \otimes |g\rangle$.
- Just before the measurement in D, the state is in general entangled (not separable):

$$|\Psi\rangle_{R_2} = U_{SM}\big(|\psi\rangle \otimes |g\rangle\big) = \big(M_g|\psi\rangle\big) \otimes |g\rangle + \big(M_e|\psi\rangle\big) \otimes |e\rangle$$

where U_{SM} is the total unitary transformation (Schrödinger propagator) defining the linear measurement operators M_g and M_e on $\mathcal{H}_S$. Since U_{SM} is unitary, $M_g^\dagger M_g + M_e^\dagger M_e = \mathbb{I}$.

The Markov chain model (3)

Just before the measurement in D, the atom/field state is:

$$M_g|\psi\rangle \otimes |g\rangle + M_e|\psi\rangle \otimes |e\rangle$$

Denote by $\mu \in \{g, e\}$ the measurement outcome in detector D: with probability $p_\mu = \langle\psi|M_\mu^\dagger M_\mu|\psi\rangle$ we get μ. Just after the measurement outcome μ, the state becomes separable:

$$|\Psi\rangle_D = \frac{1}{\sqrt{p_\mu}} \left(M_\mu|\psi\rangle\right) \otimes |\mu\rangle = \frac{\left(M_\mu|\psi\rangle\right) \otimes |\mu\rangle}{\sqrt{\langle\psi|M_\mu^\dagger M_\mu|\psi\rangle}}.$$

Markov process (density matrix formulation $\rho \sim |\psi\rangle\langle\psi|$)

$$\rho_+ = \begin{cases} \mathcal{M}_g(\rho) = \frac{M_g \rho M_g^\dagger}{\mathrm{Tr}\left(M_g \rho M_g^\dagger\right)}, & \text{with probability } p_g = \mathrm{Tr}\left(M_g \rho M_g^\dagger\right); \\ \mathcal{M}_e(\rho) = \frac{M_e \rho M_e^\dagger}{\mathrm{Tr}\left(M_e \rho M_e^\dagger\right)}, & \text{with probability } p_e = \mathrm{Tr}\left(M_e \rho M_e^\dagger\right). \end{cases}$$

Kraus map: $\mathbb{E}\left(\rho_+/\rho\right) = \mathbf{K}(\rho) = M_g \rho M_g^\dagger + M_e \rho M_e^\dagger$.

The controlled Markov chain

Control input u, state ρ (density operator), measured output y.
For the LKB photon-box we have

$$\rho_{k+1} = \begin{cases} \dfrac{M_g(u_k)\rho_k M_g^\dagger(u_k)}{\operatorname{Tr}\left(M_g(u_k)\rho_k M_g^\dagger(u_k)\right)} & y_k = g \text{ with prob. } \operatorname{Tr}\left(M_g(u_k)\rho_k M_g^\dagger(u_k)\right) \\ \dfrac{M_e(u_k)\rho_k M_e^\dagger(u_k)}{\operatorname{Tr}\left(M_e(u_k)\rho_k M_e^\dagger(u_k)\right)} & y_k = e \text{ with prob. } \operatorname{Tr}\left(M_e(u_k)\rho_k M_e^\dagger(u_k)\right) \end{cases}$$

For a general discrete-time open quantum systems:

$$\rho_{k+1} = \frac{M_\mu(u_k)\rho_k M_\mu^\dagger(u_k)}{\operatorname{Tr}\left(M_\mu(u_k)\rho_k M_\mu^\dagger(u_k)\right)} \quad y_k = \mu \text{ with prob. } \operatorname{Tr}\left(M_\mu(u_k)\rho_k M_\mu^\dagger(u_k)\right)$$

with $\mu \in \{1, \ldots, m\}$, m being the number of measure outcomes.

$$\mathbb{E}\left(\rho_{k+1}/\rho_k, u_k\right) = \mathbf{K}_{u_k}(\rho_k) = \sum_{\mu=1}^{m} M_\mu(u_k)\rho M_\mu^\dagger(u_k)$$

and $\sum_{\mu=1}^{m} M_\mu^\dagger(u_k) M_\mu(u_k) \equiv \mathbb{I}$.

Dynamical models with a precise structure

Discrete-time models are Markov chains

$$\rho_{k+1} = \frac{1}{p_\mu(\rho_k)} M_\mu \rho_k M_\mu^\dagger \quad \text{with proba.} \quad p_\mu(\rho_k) = \operatorname{Tr}\left(M_\mu \rho_k M_\mu^\dagger\right)$$

associated to Kraus maps (ensemble average, open quantum channels)

$$\mathbb{E}\left(\rho_{k+1}/\rho_k\right) = \mathbf{K}(\rho_k) = \sum_\mu M_\mu \rho_k M_\mu^\dagger \quad \text{with} \quad \sum_\mu M_\mu^\dagger M_\mu = \mathbb{I}$$

Continuous-time models are stochastic differential systems

$$d\rho = \left(-i[H,\rho] + L\rho L^\dagger - \tfrac{1}{2}(L^\dagger L\rho + \rho L^\dagger L)\right) dt + \left(L\rho + \rho L^\dagger - \operatorname{Tr}\left((L+L^\dagger)\rho\right)\rho\right) dw$$

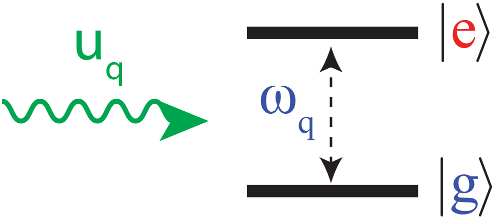
driven by Wiener processes[3] $dw = dy - \operatorname{Tr}\left((L+L^\dagger)\rho\right) dt$ with measure y and associated to Lindbald master equations:

$$\frac{d}{dt}\rho = -\tfrac{i}{\hbar}[H,\rho] + L\rho L^\dagger - \frac{1}{2}(L^\dagger L\rho + \rho L^\dagger L)$$

[3] Another common possibility not considered here: SDE driven by Poisson processes.

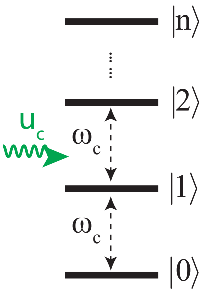
A qubit: 2 level system, 1/2 spin system

- State space: $\mathcal{H}_q = \{c_g|g\rangle + c_e|e\rangle,\ c_g, c_e \in \mathbb{C}\}$.
- Operators: $\sigma_z = |e\rangle\langle e| - |g\rangle\langle g|$, $\sigma_x = |e\rangle\langle g| + |g\rangle\langle e|$, $\sigma_y = -i|e\rangle\langle g| + i|g\rangle\langle e|$.
- Hamiltonian: $H_q = \omega_q\sigma_z/2 + u_q\sigma_x$.



Quantum harmonic oscillator

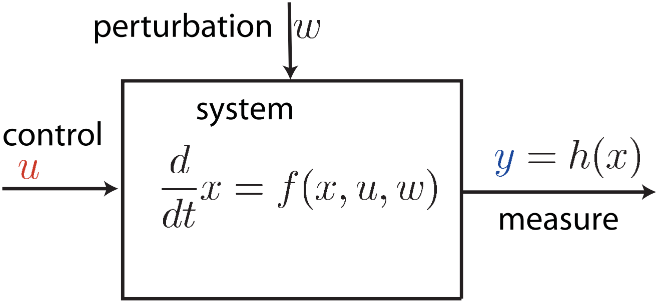
- State space: $\mathcal{H}_c = \{\sum_{n\geq 0} \psi_n |n\rangle,\ (\psi_n)_{n\geq 0} \in l^2(\mathbb{C})\}$.
- $\mathcal{D} = \{\rho \in \mathcal{L}(\mathcal{H}_c), \rho^\dagger = \rho,\ \mathrm{Tr}\,(\rho) = 1, \rho \geq 0\}$.
- Operators:
  $\mathbf{a}|n\rangle = \sqrt{n}|n-1\rangle,\ \mathbf{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle,$
  $\mathbf{N}|n\rangle = \mathbf{a}^\dagger\mathbf{a}|n\rangle = n|n\rangle,\ \mathbf{D}_\alpha = e^{\alpha\mathbf{a}^\dagger - \alpha^\dagger\mathbf{a}}.$
- Hamiltonian: $\mathbf{H}_c = \omega_c\mathbf{a}^\dagger\mathbf{a} + u_c(\mathbf{a}+\mathbf{a}^\dagger)$.
- Coherent state of amplitude $\alpha \in \mathbb{C}$:
  $|\alpha\rangle = \sum_{n\geq 0}\left(e^{-|\alpha|^2/2}\frac{\alpha^n}{\sqrt{n!}}\right)|n\rangle.$
- $\mathbf{a}|\alpha\rangle = \alpha|\alpha\rangle$.
- $\mathbf{D}_\alpha|0\rangle = |\alpha\rangle$.



Composite systems made of spins and springs

- Quantum optics, Cavity Quantum Electro-Dynamics (CQED) systems,
- Trapped ions
- Mesoscopic artificial atoms like quantum superconducting circuits
- Quantum electro-mechanics: microwave cavity optomechanics
- ...

Model of classical systems

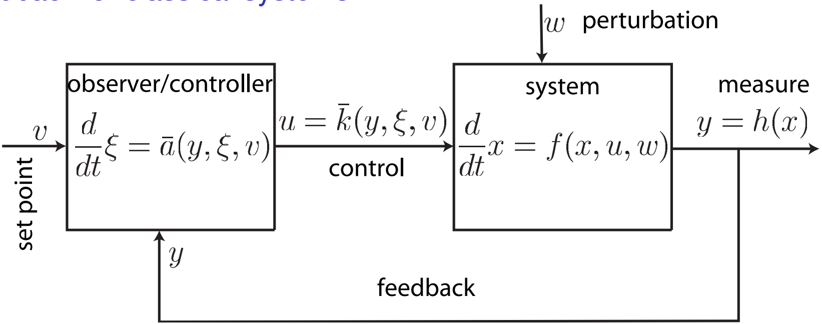


For the harmonic oscillator of pulsation ω with measured position y, controlled by the force u and subject to an additional unknown force w.

$$x = (x_1, x_2) \in \mathbb{R}^2, \qquad y = x_1$$

$$\frac{d}{dt}x_1 = x_2, \quad \frac{d}{dt}x_2 = -\omega^2 x_1 + u + w$$

Feedback for classical systems



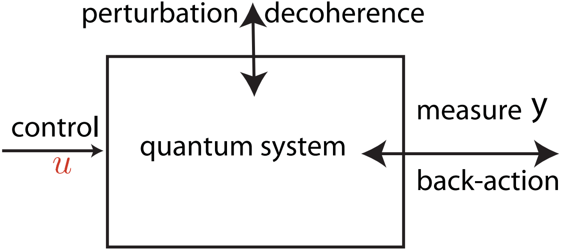
Proportional Integral Derivative (PID) for $\frac{d^2}{dt^2}y = -\omega^2 y + u + w$ with the set point v

$$u = -K_p(y-v) - K_d \frac{d}{dt}(y-v) - K_{\text{int}} \int (y-v)$$

with the positive gains $(K_p, K_d, K_{\text{int}})$ tuned as follows $(0 < \Omega_0 \sim \omega$, $0 < \xi \sim 1$, $0 < \epsilon \ll 1$:

$$K_p = \Omega_0^2, \quad K_d = 2\xi\Omega_0, \quad , K_{\text{int}} = \epsilon\Omega_0^3.$$

Feedback for the quantum system $\mathcal{S}$

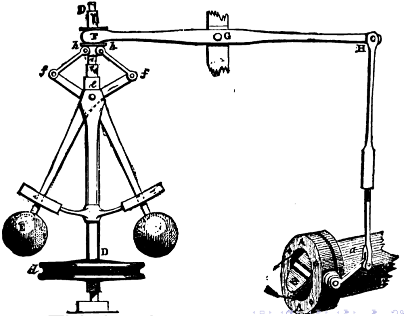


Key issue: back-action due to the measurement process.

Measurement-based feedback: measurement back-action on $\mathcal{S}$ is stochastic (collapse of the wave-packet); controller is classical; the control input u is a classical variable appearing in some controlled Schrödinger equation; u depends on the past measures.

Coherent feedback: the system $\mathcal{S}$ is coupled to another quantum system (the controller); the composite system, $\mathcal{H}_S \otimes \mathcal{H}_{controller}$, is an open-quantum system relaxing to some target (separable) state (reservoir engineering).

Watt regulator: a classical analogue of quantum coherent feedback. [4]



The first variations of speed $\delta\omega$ and governor angle $\delta\theta$ obey to

$$\frac{d}{dt}\delta\omega = -a\delta\theta$$
$$\frac{d^2}{dt^2}\delta\theta = -\Lambda\frac{d}{dt}\delta\theta - \Omega^2(\delta\theta - b\delta\omega)$$

with (a, b, Λ, Ω) positive parameters.

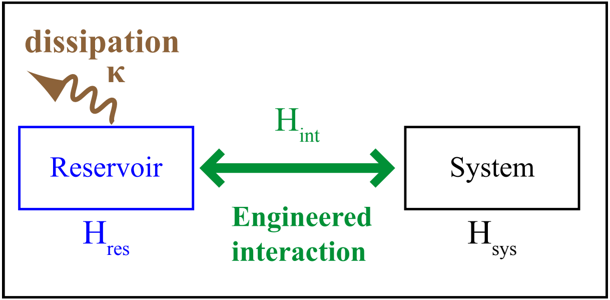
Third order system

$$\frac{d^3}{dt^3}\delta\omega = -\Lambda\frac{d^2}{dt^2}\delta\omega - \Omega^2\frac{d}{dt}\delta\omega - ab\Omega^2\delta\omega = 0$$

Characteristic polynomial $P(s) = s^3 + \Lambda s^2 + \Omega^2 s + ab\Omega^2$ with roots having negative real parts iff $\Lambda > ab$: governor damping must be strong enough to ensure asymptotic stability of the closed-loop system.

[4] J.C. Maxwell: On governors. Proc. of the Royal Society, No.100, 1868.

Reservoir Engineering[5] and coherent feedback[6]



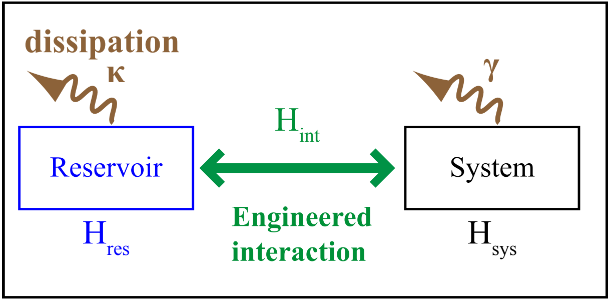
$$H = H_{res} + H_{int} + H_{syst}$$

if $\rho \underset{t\to\infty}{\to} \rho_{res} \otimes |\bar{\psi}\rangle\langle\bar{\psi}|$ exponentially on a time scale of $\tau \approx \kappa$ then . . . .

[5] Introduced by Poyatos, Cirac and Zoller, 1996.

[6] See, e.g., the lectures of H. Mabuchi delivered at the "école de physique des Houches", July 2011.

Reservoir Engineering[5] and coherent feedback[6]



$$H = H_{\text{res}} + H_{\text{int}} + H_{\text{syst}}$$

$$\ldots\ldots \quad \rho \underset{t\to\infty}{\to} \rho_{res} \otimes |\bar{\psi}\rangle\langle\bar{\psi}| + \Delta, \text{ if } \kappa \gg \gamma \text{ then } \|\Delta\| \ll 1$$

[5] Introduced by Poyatos, Cirac and Zoller, 1996.

[6] See, e.g., the lectures of H. Mabuchi delivered at the "école de physique des Houches", July 2011.

Reservoir engineering for discrete-time systems

Data: $\mathcal{H}_S$ with Hamiltonian H_S, a pure goal state $\bar{\rho}_S = |\bar{\psi}_S\rangle\langle\bar{\psi}_S|$.
Find a "realistic" meter system of Hilbert space $\mathcal{H}_M$ with initial state $|\theta_M\rangle$, with Hamiltonian H_M and interaction Hamiltonian H_{int} such that

1. the propagator $U_{S,M} = U(T)$ between 0 and time T ($\frac{d}{dt}U = -i(H_S + H_M + H_{int})U$, $U(0) = \mathbb{I}$) reads:

$$\forall |\psi_S\rangle \in \mathcal{H}_S, \quad U_{S,M}\big(|\psi_S\rangle \otimes |\theta_M\rangle\big) = \sum_\mu \big(M_\mu |\psi_S\rangle\big) \otimes |\lambda_\mu\rangle$$

   where $|\lambda_\mu\rangle$ is a ortho-normal basis of $\mathcal{H}_M$.
2. the resulting measurement operators M_μ admit $|\bar{\psi}_S\rangle$ as common eigen-vector, i.e., $\bar{\rho}_S$ is a fixed point of the Kraus map $\mathbf{K}(\rho) = \sum_\mu M_\mu \rho M_\mu^\dagger$: $\quad \mathbf{K}(\bar{\rho}_S) = \bar{\rho}_S$.
3. iterates of $\mathbf{K}$ converge to $\bar{\rho}_S$ for any initial condition ρ_0:

$$\lim_{k \mapsto +\infty} \rho_k = \bar{\rho}_S \text{ where } \rho_k = \mathbf{K}(\rho_{k-1}).$$

Here the reservoir is made of the infinite set of identical meter systems with initial state $|\theta_M\rangle$ at $t = (k-1)T$ and interacting with $\mathcal{H}_S$ during $[(k-1)T, kT]$, $k = 1, 2, \ldots$.

Quantum filtering and estimation

- Discrete-time systems: knowing the input u and the measure outcome y until time step k, the estimate ρ_k^{est} of ρ_k is given recursively by

$$\rho_{k+1}^{\text{est}} = \frac{1}{\text{Tr}\left(M_{y_k}(u_k)\rho_k^{\text{est}}M_{y_k}^\dagger(u_k)\right)} M_{y_k}(u_k)\rho_k^{\text{est}}M_{y_k}^\dagger(u_k).$$

- Continuous-time systems: knowing the input u and the measure outcomes y until time t, the estimate ρ_t^{est} of ρ_t is given recursively by

$$d\rho_t^{\text{est}} = \left(-i[H, \rho_t^{\text{est}}] + L\rho_t^{\text{est}}L^\dagger - \frac{1}{2}(L^\dagger L\rho_t^{\text{est}} + \rho_t^{\text{est}}L^\dagger L)\right) dt$$
$$+\left(L\rho_t^{\text{est}} + \rho_t^{\text{est}}L^\dagger - \text{Tr}\left((L + L^\dagger)\rho_t^{\text{est}}\right)\rho_t^{\text{est}}\right)(dy_t - \text{Tr}\left((L + L^\dagger)\rho_t^{\text{est}}\right) dt)$$

since $dw_t = dy_t - \text{Tr}\left((L + L^\dagger)\rho_t\right) dt$.

Open issues: convergence and robustness analysis of such non-linear filters.

And many other subjects

Mathematical **quantum** systems theory taking into account

- composite systems based on tensor-product
- measurement back-action
- non-commutative computations with operators
- infinite dimension (harmonic oscillator, transmission line, delays,. . .)

to investigate on physical examples

- stability and estimation of convergence rates,
- dissipation and passivity
- model reduction, perturbations methods, multiple scales
- controllability, motion planing, optimal control and stabilizing feedbacks
- observability, filtering, parameter estimations, optimal tomography
- input/ouput modelling, realization, quantum circuits
- . . .